CSUC
Department of Physics
301A Mechanics
***PORTFOLIO* Problem 3.a)** due: End of Exam Week !

I. PERTURBATION ON A PLANETARY ORBIT

The effect (as we will deduce later) of a uniform distribution of "dust" about the sun of density ρ is to introduce an additional very small attractive central "dust potential" $U_{dust} = \frac{1}{2}\left(m\frac{4}{3}\pi\rho G\right) r^2$. Let us now add this to the Kepler potential from the sun to better describe the environment our planets actually find themselves in. This is a perturbation problem! The additional potential is "small" and makes everything *shift* just a little. Once again, the idea is to observe what happens in successive "consistent order of smallness". This is, of course, a Taylor series expansion.

<u>Problem 1.</u>) The mass of the sun, M, produces the Kepler potential we are so familiar with. When we add in the "dust potential" the situation is still central and angular momentum is conserved. Find the angular velocity of revolution of the planet in a circular orbit of radius r_o first, and then go on to find the *angular frequency* of small radial oscillations *about* r_o. Hence, show that, if U_{dust} is much smaller than the Kepler contribution from the sun (as we now suppose) then, the *nearly circular* orbits will approximately be ellipses with major axes precessing slowly at angular velocity:

$$\omega_{prec} = 2\pi\rho\left(\frac{r_o^3 G}{M}\right)^{1/2}$$

<u>Problem 2.</u>) Does the axis precess in the same or opposite direction to the orbital angular velocity ? Look up M and the radius of the orbit of Mercury, and then calculate the density of dust required to cause a precession of 41 seconds of arc per century, as is actually observed in careful modern measurements.

1. *General reflections on Problem P3*

This is a very sweet little problem. It is actually authentic, i.e. people *actually* use it. What is of major importance here, however, is to realize that very many problems proceed <u>just like this one</u>. The full solution would be impossible to find. However, we don't need the full solution, but just an approximation that is highly numerically accurate.

Insight #1) Any perturbation introduces a new small "size" parameter (here we will call it λ). Everything (and every equation) will be expanded in powers of it. We will collect terms of like size and these form our new working equations.

Insight #2) Always express the features of a problem in terms of "itself"! This will mean that you use <u>*Natural Units*</u> as an essential technique. Only then can we discern the essential relationships clearly and without being distracted by a cloud of confusing constants. Proceed slowly and watch how the pieces fit together - you will come to really enjoy how this works.

2. *How to do it: ... Start with the <u>unperturbed</u> Kepler problem, as usual (i.e. the Zeroth Problem).*

Step #1) Write down the two conservation equations, energy and angular momentum for the Kepler problem, as we always do. Then, as usual, we remove $\dot{\theta}$ in the energy equation in favor of r by use of the angular momentum equation. Next, we substitute out the **time** as the independent variable in favor of the angle variable θ by again using the angular momentum relation. Recall that, next, we gather "potential like" terms into an *effective* potential.

Step #2) Proceed to find the minimum of the effective potential which then identifies our Natural Units of length and energy: $\{L, \mathcal{E}\}$. Now make sure that all lengths are scaled against the Natural Length Unit and, *Voilà*, the equation simplifies enormously. We quickly observe that is is much more natural to use the variable $u \equiv L/r$.

3. ***Prepare to add in the Perturbation.***

Step #3) We observe that the cluster of constants in our perturbation potential has the net dimension and form of a "perturbation spring constant". That is, we may define: $k_p \equiv \left(m\frac{4}{3}\pi\rho G\right)$ so that $U_{dust} = \frac{1}{2}\,k_p\,r^2$. In fact, we may take this one small step further by scaling r against our unit of length, in which case we achieve:

$$U_{dust} = \frac{1}{2}\,k_p \cdot L^2\ (r/L)^2 = \frac{1}{2}\,k_p \cdot L^2\ \cdot u^{-2}$$

If we now add this in to our conservation of energy equation we observe that, upon dividing the entire equation through by the unit of energy $\mathcal{E}$, we now have our natural *dimensionless small parameter*, namely $\lambda \equiv k_p \cdot L^2\,/\mathcal{E}$, ... the natural ratio of two energies. At this point we start the problem over again.

4. ***Examine the new "Effective Potential".***

Our new effective potential has <u>three</u> terms. We assert that the equilibrium point has *nudged* just a bit and that the curvature at the equilibrium has also changed just a bit. We want to evaluate these changes to <u>*first order*</u> in the very small dimensionless parameter λ. So here we go.

Step #4) Take the new effective potential and find its minimum value corrected to first order in λ. This will yield a new u_{min} value at which the potential finds its minimum. Now perform a Taylor expansion of $U_{effective}$ about that new minimum value correct to quadratic terms in $(u - u_{min})$ as we always do. Find the coefficient of the quadratic term and its first order change from what it was before (remember the fist order term will vanish by dint of expanding around the minimum). The equation sitting before you describes the dependence of u on θ. The major question is whether the radial motion completes one cycle of its motion as θ sweeps out one circle of angle. If that is **not** so, ... then this is **not** a closed orbit. *Precession!* All the questions of this PORTFOLIO problem are answered by examining this last relationship. Proceed!